M434 Differential geometry

Level: 3 Credit points: 30
No computer required

Summary

We develop the interactions between linear algebra and differential calculus to study the main ideas and examples of classical differential geometry, using modern mathematical techniques where they simplify or clarify the exposition. All the curves and surfaces studied are in three-dimensional Euclidean space and so can be readily visualized and described explicitly in terms of standard functions. The course is based on the set book Elementary Differential Geometry by B. O'Neill (Academic Press).

Description

The course develops the interaction between linear algebra and differential calculus in order to study the main ideas and examples of classical differential geometry, using modern mathematical techniques where they simplify or clarify the exposition. All the curves and surfaces studied are in three-dimensional Euclidean space and so can be readily visualized and explicitly described in terms of standard functions. The course is based on the set book, with a combined course guide and reminder of the mathematics you are expected to be already familiar with.

The six parts of the course cover the first six chapters of the set book. *Chapter I* extends the basic ideas of linear algebra and differential calculus to deal with vector fields, directional derivatives, differential forms and derivative mappings. An additional section deals with the chain rule.

Chapter II studies curves in terms of their tangent, normal and binormal vector fields. This Frenet frame leads to definitions of the curvature and torsion of a curve. Techniques for calculating these are developed and they are shown to reflect the geometry.

Chapter III studies mappings that preserve distance - isometries - and shows that if two curves have the same speed, curvature and torsion then there exists an isometry mapping one to the other.

Chapter IV shows how surfaces can be described in terms of co-ordinate patches. Typical surfaces are the sphere, the torus and the helicoid. Tangent vectors are constructed and used to describe calculus on a surface.

Chapter V uses the calculus to define the shape operator of a surface. This is a linear mapping on tangent vectors and can be represented by a 2 ´ 2 matrix. This matrix is used to define the Gaussian curvature (the determinant) and the mean curvature (the trace), and the principal directions (the eigenvectors). Formulae for calculating these are found and their relationship with the geometry of the surface and special curves (principal curves, asymptotic curves and geodesics) is discussed.

Chapter VI is about which properties of a curve are intrinsic, that is, which depend on the internal structure of the surface and not on how we describe them as subspaces of a larger space. We show that Gaussian curvature is such a property, and obtain an alternative method of calculating it.

Entry

The course assumes that you have already met certain basic mathematical concepts and techniques. Some of these are briefly reviewed in the course, but it is still advisable to have some familiarity with them before you begin. They are:

• Vector spaces, especially R2 and R3. Bases and co-ordinates.

• Dot (or scalar) product; lengths and angles. Orthogonality. Orthonormal bases, orthonormal

expansion.

• Linear transformations between vector spaces. Representation of a linear transformation by a matrix.

• Determinants. Calculation of determinants of 3 ´ 3 matrices.

• Cross (or vector) product.

• Definition of characteristic values and characteristic vectors (eigenvalues and eigenvectors) of a linear transformation (of a vector space to itself).

• Isometries: translations, rotations, reflections. Orthogonal matrices.

• Standard functions: polynomials, exp, log, cos, sin, tan, cosh, sinh. Properties of these functions and functions made up from them. Graphs of such functions. Differentiation of such functions.

• Trigonometric identities, especially cos2x + sin2x = 1. Corresponding identities for cosh and sinh: cosh2x - sinh2x = 1.

• Integration of simple functions.

• Partial differentiation. Calculation of partial derivatives of functions of two or three variables. (There is a brief review of these ideas in the course guide.)

You could get all this from the Level 2 courses M203 *Introduction to pure mathematics* and MST207 *Mathematical methods, models and modelling* (or the earlier course MST204 *Mathematical models and methods*). Your Regional Centre will be able to tell you where you can look at reference copies, or you can buy selected materials from Open University Worldwide Ltd.

This is a Level 3 course, which makes intellectual demands appropriate to the final year of an honours degree. If you have any doubt about your ability to take the course please seek advice from your Regional Centre..

Preparatory work

If you would like to acquaint yourself with some of the main elements of the course, the first chapter of the set book gives a good introduction.

If you are disabled

The course is not recommended if you have impaired sight, because diagrams in the set book are used in the assignments and recordings of printed course materials are not available at present. If you are a new student, make sure that you have our booklet *Meeting Your Needs*.

What's included

Course books, other printed materials.

Set book to buy

B. O'Neill *Elementary Differential Geometry*, (2nd edition), Academic Press, £34.95 (hardback)

Tuition and counselling

You will have a tutor who will help you with the course material and mark and comment on your written work, and whom you can ask for advice and guidance. We may also be able to offer group tutorials or day-schools that you are encouraged, but not obliged, to attend. Where your tutorials are held will depend on the distribution of students taking the course. Ask your Regional Centre if you need to know more before you decide whether to register. Your Regional Centre will provide a counselling service to give you general help with your studies.

Assessment

There are four tutor-marked assignments and an examination. Assessment is an essential part of the teaching, so you are expected to complete it all. But if you unavoidably miss or do badly in an assignment some courses allow you a 'substitution score', calculated as a weighted average of all your scores for the course. In M434 this rule can apply to one assignment. You will be given more detailed information when you begin the course.

Qualifications

This is a specified course in our MMath degree, and in our honours BSc in Mathematical Sciences and Computing, and in Mathematical Sciences. It can also count towards most of our other degrees at bachelor's level - it is equally appropriate to a BA or a BSc - and towards our discontinued Advanced Diploma in Mathematics Education. It is up to you to ensure that you are properly informed about the circumstances in which the course can count towards these qualifications.

Professional recognition

This course can help you to gain recognition from a professional body. Ask your Regional Centre for *Recognition* leaflets 3.3 and 3.6.

For the future

This course is presented every four years, and 2001 may be its last year.

TEACHING RESOURCES BRIEFING

M434 DIFFERENTIAL GEOMETRY

An introduction to the ideas and methods of differential geometry by studying curves and surfaces in three-dimensional space. Attention is restricted to three dimensions. Nevertheless, the ideas and basic techniques developed here do generalize and will provide a good foundation should further reading in the subject be required.

LEVEL: **Fourth level**

All prices shown exclude VAT and carriage

STUDY UNITS

Course Guide and Introduction

Margolis, B.

Coverage: Introduction • Course guide • Mathematical prerequisites.

1993 (16pp) 0 7492 4769 X M434 0 £6.45

Calculus on Euclidean Space

Margolis, B.

Coverage: Introduction • Euclidean space • Tangent vectors • Directional derivatives • Curves • 1-Forms • Differential forms • Mappings.

1993 (36pp) 0 7492 4770 3 M434 I £6.45

Frame Fields and Curves

Margolis, B.

Coverage: Introduction • Dot product • Curves • The Frenet formulas • Arbitrary-speed curves • Covariant derivatives • Frame fields • Connection forms • The structural equations.

1993 (36pp) 0 7492 4771 1 M434 II £6.45

Euclidean Geometry

Margolis, B.

Coverage: Introduction • Isometries • Derivative maps of isometries • Orientation • Euclidean geometry • Congruence of curves.

1993 (24pp) 0 7492 4772 X M434 III £6.45

Calculus of a Surface

Margolis, B.

Coverage: Introduction • Surfaces • Patch computations • Functions and tangent vectors • Differential forms on a surface • Mappings of surfaces • Topological properties • Summary • Appendix: integration.

1993 (40pp) 0 7492 4773 8 M434 IV £6.45

Shape Operators

Margolis, B.

Coverage: Introduction • The shape operator • Normal curvature • Gaussian curvature • Computational techniques • Special curves • Surfaces of revolution • Summary.

1993 (28pp) 0 7492 4774 6 M434 V £6.45

Geometry of Surfaces

Margolis, B.

Coverage: Introduction • The fundamental equations • Form computations • Orthogonal coordinates • The Gauss map.

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M434

Assignment Booklet

Contents

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2	TMA M434 01 (Sections I.1–II.2 of *O'Neill*)	27 March 2001
5	TMA M434 02 (Sections II.3–III.5 of *O'Neill*)	29 May 2001
8	TMA M434 03 (Sections IV.1–V.3 of *O'Neill*)	24 July 2001
11	TMA M434 04 (Sections V.4–VI.2 and VI.6 of *O'Neill*)	18 September 2001

Instructions for TMAs

To save repetition, all *O'Neill*'s conventions will be used in TMA questions and may be used by you, without comment, when writing your solutions.

The usual way to write symbols that would be printed in bold is to underline them.

If a question says 'write down ...', just the answer will be accepted. In all other cases, there are specific marks awarded for your working. We recommend showing your working for *all* questions. This will give your tutor the opportunity to award you some marks for a question where you have the working partially correct, even though you may not obtain the correct final answer.

For each TMA, please send your answers to all the questions to your tutor, along with an appropriately completed PT3 form. Note that all questions are used for assessment.

You will find instructions on how to fill in the PT3 form in the *Student Handbook*. Be sure to fill in the correct Assignment Number, e.g. for the first TMA this is

$\boxed{\text{M434 01}}$.

Do not forget that the cut-off date for a TMA represents the latest date by which your tutor should receive your answers, unless you have special permission, so allow for the time your answers will take to go through the post.

SUP 54264 7
3.1

You should be able to answer Question 1 after you have studied Sections 1–5 of Part I.

Question 1 (25 marks)

The function $f : \mathbf{E}^3 \longrightarrow \mathbf{R}$, the point $\mathbf{p}$, the vector field V and the curve α are defined as follows:

$$f = xy + z^2,$$
$$\mathbf{p} = (-2, 1, 3),$$
$$V = x^2 U_1 + xyU_2 - zU_3,$$
$$\alpha(t) = (-2\cos t + 4\sin t, \cos t - 2\sin t, 3(\cos t - \sin t)), \quad t \in \mathbf{R}.$$

The tangent vector $\mathbf{v_p}$ at $\mathbf{p}$ is defined by

$$\mathbf{v_p} = V(\mathbf{p}).$$

(i) Calculate the partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial z},$$

and evaluate them at $\mathbf{p}$. [4]

(ii) Write down $\mathbf{v_p}$, and hence find, *using the definition*, the directional derivative $\mathbf{v_p}[f]$. [5]

(iii) Use the result of part (i) and Lemma I.3.2 of *O'Neill* to check your answer to part (ii). [3]

(iv) Express the directional derivative $V[f]$ in terms x, y and z. By making a suitable substitution, give an alternative calculation of $\mathbf{v_p}[f]$. Your answer should include a brief explanation of why you made the substitution that you did. [4]

(v) Find the composite

$$f(\alpha(t)).$$

Calculate the value of the derivative

$$(f(\alpha(t)))'$$

at $t = 0$. Compare your answer with that of part (ii), and explain the result of the comparison in terms of results proved in *O'Neill*. [5]

(vi) Find the differential df of f, expressing your answer in terms of the 1-forms dx, dy and dz. Hence find

$$df(V).$$

Explain, briefly, the relationship between your answers to this part and to part (iv). [4]

You should be able to answer Questions 2 and 3 after you have completed your study of Part I.

Question 2 (18 marks)

The function $F : \mathbf{E}^3 \longrightarrow \mathbf{E}^3$ and the function $G : \mathbf{E}^3 \longrightarrow \mathbf{E}^2$ are defined by

$$F = (x + y, y^2, yz),$$
$$G = (xz, yz).$$

(i) Find the Jacobian matrices representing the derivative maps F_* and G_*. [3]

(ii) Find the composite function $H = G(F)$ from $\mathbf{E}^3$ to $\mathbf{E}^2$. [2]

(iii) Evaluate the matrix representing G_* at F. [2]

(iv) Use the composite rule to find the Jacobian matrix representing the derivative map H_*. Confirm your result by direct calculation. [4]

(v) Find the set of points in $\mathbf{E}^3$ for which F is *not* regular. [4]

(vi) Find the image of the xz plane under F. [3]

Question 3 (12 marks)

The function $f : \mathbf{E}^3 \longrightarrow \mathbf{R}$, the 1-form ϕ and the 2-form η are defined by

$$f = xy - z^2,$$
$$\phi = y\,dx - x\,dy + z\,dz,$$
$$\eta = z\,dx\,dy + x^2\,dy\,dz + y\,dx\,dz.$$

Calculate the following.

(i) $df \wedge \phi$ [3]

(ii) $d\phi$ [2]

(iii) $d(f\phi)$ [2]

(iv) $d\eta$ [3]

(v) $df \wedge \eta$ [2]

You should be able to answer Question 4 after you have studied Section 1 of Part II.

Question 4 (17 marks)

The tangent vectors

$$\mathbf{e}_1 = (\ 1,\ 2,\ 2)/3,$$
$$\mathbf{e}_2 = (\ 2, -2,\ 1)/3,$$
$$\mathbf{e}_3 = (-2, -1,\ 2)/3$$

are all based at a point $\mathbf{p}$ of $\mathbf{E}^3$.

(i) Write down the attitude matrix, A, of this set of vectors. [1]

(ii) Show that $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ form a frame at p [4]

(iii) Write down the inverse A^{-1} of A. Explain briefly why your answer is correct. [2]

(iv) Evaluate $\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3$, and hence determine whether the frame is right- or left-handed. [3]

(v) Express the tangent vector $\mathbf{v_p} = (3, -6, 9)_\mathbf{p}$ at $\mathbf{p}$ as a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. [4]

(vi) Calculate $(\mathbf{e}_1 + 2\mathbf{e}_2 - 3\mathbf{e}_3) \cdot (3\mathbf{e}_1 - \mathbf{e}_2 + 7\mathbf{e}_3)$. [3]

You should be able to answer Question 5 after you have studied Section 2 of Part II.

Question 5 (16 marks)

The curve α is defined by

$$\alpha(t) = (\cos(e^t - 1), \sin(e^t - 1), e^t - 1), \quad t \in \mathbf{R}, \quad -1 < t < 1.$$

(i) (a) Find the velocity and speed of α at $\alpha(t)$. [3]

(b) Find the arc-length function $s(t)$ for α, based at $t = 0$. [3]

(c) Find a unit-speed reparametrization β of α, and confirm, by direct calculation, that β is unit-speed. [5]

(ii) Suppose that f is a one-one function from a set $D \subseteq \mathbf{R}$ to $\mathbf{R}$, that $0 \in \mathbf{R}$, that $f(0) = 0$ and that γ is a curve defined by

$$\gamma(t) = (\cos(f(t)), \sin(f(t)), f(t)), \quad t \in D.$$

Find a unit-speed reparametrization of γ. [5]

Question 6 is concerned with differential forms. It makes use of the explanation of the action of a wedge product of 1-forms on a pair of tangent vectors as discussed in the Commentary for Section I.6.

Question 6 (12 marks)

(i) The functions ϕ and ψ from the set of all tangent vectors at all points in $\mathbf{E}^3$ to the real numbers are defined, with the usual notation, by

$$\phi(\mathbf{v_p}) = 2\mathbf{v} \cdot 3\mathbf{p},$$
$$\psi(\mathbf{v_p}) = v \cdot v.$$

(a) Exactly one of ϕ and ψ is a 1-form. Decide which is, and which is not, a 1-form, justifying your answers. [4]

(b) For the function which is a 1-form, evaluate it on U_1, U_2 and U_3, and hence express it in terms of the 1-forms dx, dy and dz. [3]

(ii) The 1-form ζ on $\mathbf{E}^3$ is defined by

$$\zeta = z\,dx + x\,dy + y\,dz.$$

Show that if $\mathbf{v_p}$ and $\mathbf{w_p}$ are tangent vectors at $\mathbf{p}$, then

$$d\zeta(\mathbf{v_p}, \mathbf{w_p}) = (dx + dy + dz)((\mathbf{v} \times \mathbf{w})_\mathbf{p}).$$ [5]

You should be able to answer Question 1 after you have studied Section 3 of Part II.

Question 1 (20 marks)

The curve α, with Frenet apparatus T, N, B, κ and τ, is defined by

$$\alpha(t) = \tfrac{1}{3}(2\cos t - \sin t, \cos t + 2\sin t, 2t), \quad t \in \mathbf{R}.$$

(i) Show that α is unit-speed. [4]

(ii) Find the Frenet apparatus of α. [12]

(iii) Explain what the results obtained in part (ii) tell you about the curve α. [4]

You should be able to answer Question 2 after you have studied Sections 3–4 of Part II.

Question 2 (20 marks)

The curve α, with Frenet apparatus v, T, N, B, κ and τ, is defined by

$$\alpha(t) = (2\cosh t, \sinh t, 2t), \quad t \in \mathbf{R}.$$

(i) Find the Frenet apparatus of α. [15]

(ii) Find the Frenet frame for α at $\alpha(0)$. [3]

(iii) Find the vector

$$\mathbf{u} = (T + 2B)/\sqrt{5},$$

and hence show, directly from the definition, that α is a cylindrical helix. State, with a brief reason, whether or not α is a circular helix. [2]

You should he able to answer the remaining questions when you have studied as far as Section 5 of Part III.

Question 3 (20 marks)

(i) This part concerns the curve β, with Frenet apparatus v_β, T_β, N_β, B_β, κ_β and τ_β, and also the curve α that you investigated in Question 2 ahove. You will need the results from Question 2.

The speed and curvature of β are the same as tbose of α, the torsion of β is minus tbe torsion of α, $\beta(0) = (2, 2, -1)$ and the Frenet frame of β at $t = 0$ is

$$T_\beta(0) = (1, 0, -1)/\sqrt{2},$$
$$N_\beta(0) = (1, 0, 1)/\sqrt{2},$$
$$B_\beta(0) = (0, 1, 0).$$

(a) Explain why α and β are congruent. [1]

(b) Construct an isometry F sucb that $F(\alpha) = \beta$. You should give F in the form TC, specifying both the orthogonal part C and tbe translation part T. [7]

(c) Write down $\beta(t)$ in terms of t. [2]

(ii) For each of the following mappings $F : \mathbf{E}^3 \longrightarrow \mathbf{E}^3$, state whether or not it is an isometry of $\mathbf{E}^3$, and give a reason for your answer.

(a) $F = (x, x, y)$ [2]

(b) $F = (x^2, y^2 + 1, z^2)$ [2]

(c) $F = ((x - z)/\sqrt{2}, (x + y + z)/\sqrt{3}, (x - 2y + z)/\sqrt{6})$ [2]

(d) $F = (x + 1, y - x, 4 - z)$ [2]

(e) $F = (2 - x, 3 - y, 4 - z)$ [2]

Question 4 (10 marks)

A family of curves α_k, $k \in \mathbf{R}$, is defined as follows:

$$\alpha_k(t) = (\cos t, \sin t, kt), \quad t \in \mathbf{R}.$$

The curvature and torsion functions of α_k are

$$\kappa_k = \cancel{\frac{\sqrt{1+k^2}}{k^3}}, \quad \kappa_k = \frac{1}{1+k^2}$$

$$\tau_k = \frac{k}{1+k^2}.$$

(i) Establish tbe following results.

(a) All curves in tbe family are circular helices. [2]

(h) Exactly one curve in tbe family is a plane curve. [2]

(c) If α_k is *not* planar, then α_k is congruent to exactly one other curve in the family. [3]

(ii) If α_k is not planar, find the isometry whose existence is guaranteed by the result in part (i)(c) above. (You sbould not need to calculate any Frenet frames for tbis part.) [3]

Question 5 (10 marks)

The vector fields V and W and the function f are defined by

$$V = xyU_1 + zU_2 - y^2U_3,$$
$$W = z^2U_1 - (xyz)U_3,$$
$$f = xy - z^2.$$

(i) Calculate $V[f]$. [3]

(ii) Calculate $\nabla_V W$. [4]

(iii) Calculate $\nabla_V fW$. [3]

Question 6 (10 marks)

The vector fields F_1, F_2 and F_3 are defined by

$$F_1 = (\cos f)\, U_1 - (\sin f)\, U_3,$$
$$F_2 = (\sin f)\, U_1 + (\cos f)\, U_3,$$
$$F_3 = -U_2,$$

where f is a differentiable function from $\mathbf{E}^3$ to $\mathbf{R}$.

(i) Show that F_1, F_2 and F_3 form a frame field. [3]

(ii) Find the dual 1-forms for F_1, F_2 and F_3, giving your answer in terms of the differentials dx, dy and dz. [2]

(iii) Find the matrix ω of connection forms for F_1, F_2 and F_3, giving your answer in terms of the differential df of f. [5]

Question 7 (10 marks)

The curve α is a unit-speed circular helix, with Frenet apparatus T, N, B, $\kappa > 0$ and τ.

The curve β is defined by

$$\beta(t) = \alpha(t) + T(t),$$

with Frenet apparatus v_β, T_β, N_β, B_β, κ_β and τ_β.

(i) Find κ_β and τ_β in terms of κ and τ. [8]

(ii) Show that β is a plane curve if and only if α is a plane curve. [2]

You should be able to answer Question 1 after you have studied Sections 1–3 of Part IV.

Question 1 (25 marks)

This question concerns the function $f : \mathbf{E}^3 \longrightarrow \mathbf{R}$ defined by

$$f = -x^2 + y^2 + 2z$$

and the mapping $\mathbf{x} : \mathbf{E}^2 \longrightarrow \mathbf{E}^3$ defined by

$$\mathbf{x}(u, v) = (u + v, u - v, 2uv).$$

The set of points M is defined by

$$M : f = 0.$$

(i) Show that M is a surface in $\mathbf{E}^3$. [3]

(ii) Show that $\mathbf{x}$ is a coordinate patch. [5]

(iii) Show that $\mathbf{x}$ covers the whole of M, by carrying out the following steps.

(a) Show that the image $\mathbf{x}(u, v)$ lies in M for all values of u and v.

(b) Show that each point $\mathbf{p}$ of M can be expressed in the form

$$\mathbf{p} = \mathbf{x}(u_0, v_0)$$

for suitable values of u_0 and v_0. [5]

(iv) (a) Find values of u_0 and v_0 such that

$$(1, 5, -12) = \mathbf{x}(u_0, v_0).$$

(b) Show that the tangent vector

$$\mathbf{v_p} = (5, 1, 0)_{(1,5,-12)}$$

is tangent to the surface M. [4]

(v) Let g be the function defined on M by

$$g = x + 2y - z^2.$$

(a) Express g as a function of u and v.

(b) Show that

$$\mathbf{x}_u(u, v)[g] = 3 - 8uv^2,$$

and find a similar expression for

$$\mathbf{x}_v(u, v)[g].$$ [5]

(vi) Calculate

$$\mathbf{v_p}[g],$$

where $\mathbf{v_p}$ is as defined in part (iv). [3]

You should be able to answer Question 2 after you have studied Section 5 of Part IV.

Question 2 (25 marks)

This question concerns the surface of revolution M parametrized by

$$\mathbf{x}(u, v) = ((3 + \sin u)\cos v, (3 + \sin u)\sin v, \cos u),$$

the sphere S parametrized by

$$\mathbf{y}(u, v) = (\sin u, \cos u \cos v, \cos u \sin v)$$

and the mapping F from M to S defined by

$$F : \mathbf{x}(u, v) \longmapsto \mathbf{y}\left(\frac{\pi}{2} - u, \frac{\pi}{2} + v\right).$$

(i) Find the partial velocities on both M and S. [4]

(ii) Construct unit normal vector fields U on M and V on S. Your answers should be in the form

$$U(\mathbf{x}(u, v)) = \ldots$$

etc. [4]

(iii) Find $F_*(\mathbf{x}_u(u, v))$ and $F_*(\mathbf{x}_v(u, v))$, expressing your answers in terms of $\mathbf{y}_u$ and $\mathbf{y}_v$ at a suitable point of S. [5]

The 2-form A on M is defined by

$$A(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1 \times \mathbf{v}_2 \cdot U(\mathbf{p}),$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are tangents to M at $\mathbf{p}$.

The 2-form B on S is defined by

$$B(\mathbf{w}_1, \mathbf{w}_2) = \mathbf{w}_1 \times \mathbf{w}_2 \cdot V(\mathbf{p}),$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are tangents to S at $\mathbf{p}$.

(iv) Evaluate

(a) $A(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v))$,

(b) $B(\mathbf{y}_u(u, v), \mathbf{y}_v(u, v))$. [6]

(v) Calculate

$$(F^*B)(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)),$$

and hence show that

$$F^*B = g(u, v)A,$$

where $g(u)$ is a function of u only. Write down g. [6]

You should be able to answer Questions 3 and 4 after you have Section 3 of Part V.

Question 3 (25 marks)

The surface M is parametrized by the mapping

$$\mathbf{x}(u,v) = (\sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v, \sinh^{-1} u), \quad u, v \in \mathbf{R}.$$

(i) Find tbe partial velocities, and hence construct a unit normal vector field U on M. [4]

Let S be the shape operator on M derived from U.

(ii) Working from Definition V.1.1 of *O'Neill*, find

$$S(\mathbf{x}_u(u,v)) \quad \text{and} \quad S(\mathbf{x}_v(u,v)),$$

giving your answers in terms of $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$. Hence find tbe normal curvatures in the directions of the partial velocities. [7]

(iii) Write down the matrix representing S witb respect to the basis of partial velocities $\{\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)\}$ at $\mathbf{x}(u,v)$. Hence find the Gaussian and mean curvature functions on M. [3]

(iv) Write down the principal curvatures of M at $\mathbf{x}(u,v)$, giving a brief reason for your answers. Does M have any umbilic points? Justify your answer briefly. [4]

(v) Write down the eigenvectors of S at $\mathbf{x}(u,v)$, and verify directly that they are orthogonal. [4]

(vi) How many asymptotic directions are there at each point of M? Justify your answer briefly. [3]

Question 4 (25 marks)

Let β be a unit-speed curve with Frenet apparatus T, N, B, $\kappa > 0$ and τ, where the torsion is *constant*. Tbe surface M is parametrized by the mapping

$$\mathbf{x}(u,v) = \beta(u) + vB(u),$$

where u and v are suitably restricted so that $\mathbf{x}$ does parametrize M.

(i) Find the partial velocities for this parametrization, and hence express a normal vector field U on M in terms of the parameters and the Frenet apparatus of β. [5]

Let S be the shape operator on M derived from U.

(ii) Calculate

$$S(\mathbf{x}_u(u,v)) \quad \text{and} \quad S(\mathbf{x}_v(u,v)),$$

giving your answers in terms of $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$. Hence find the matrix representing S with respect to the partial velocities. [6]

(iii) Write down the Gaussian curvature, K, and the mean curvature, H, for M. [3]

(iv) For each of the attributes 'flat' and 'minimal', is it possible for M to have that attribute? If not, explain why not; if so, explain any necessary restrictions on the nature of the curve β. [5]

(v) (a) Find the principal curvatures for M at $\mathbf{x}(u,v)$. You need not simplify your expressions. [2]

(b) Is it possible for M to have umbilic points? If so, find them all; if not, explain why not. [4]

Your attention is drawn to Section 9.6 of the *Student Handbook*, which reads as follows: 'So that scores can he recorded and documentation prepared at the end of each course, *no extension to the cut-off date will normally be allowed for a course's last assignment.*'

You should he able to answer Question 1 after you have studied Sections 4–5 of Part V.

Question 1 (25 marks)

The surface M is parametrized by the mapping

$$\mathbf{x}(u,v) = (v \sinh u, v \cosh u, v), \quad v > 0, \quad u, v \in \mathbf{R}.$$

(i) Find the partial velocities, and hence construct a unit normal vector field U on M. [4]

(ii) Using the field U constructed in part (i), find the functions E, F, G, l, m and n on M. [7]

(iii) Find the Gaussian, mean and principal curvature functions on M. [6]

(iv) How many asymptotic directions does M have at each point? Justify your answer. [3]

(v) Are the u and v parameter curves geodesics in M? Justify your answer. [5]

You should be able to answer Question 2 after you have studied to the end of Part V.

Question 2 (20 marks)

A surface M is parametrized by the Monge patch

$$\mathbf{x}(u,v) = (u, v, f(u,v))$$

for a function $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$. The curve α in M has coordinate functions α_1 and α_2, that is,

$$\alpha(t) = \mathbf{x}(\alpha_1(t), \alpha_2(t)).$$

(i) Express a unit normal vector field U on M in terms of the partial derivatives f_u and f_v of f. [5]

(ii) Express the velocity α' of α in terms of the derivatives of α_1, α_2 and f, and hence show that

$$\alpha'' = (\alpha_1'', \alpha_2'', f_{uu}\alpha_1'^2 + 2f_{uv}\alpha_1'\alpha_2' + \overset{f_{vv}}{\cancel{f_v}}\alpha_2'^2 + f_u\alpha_1'' + f_v\alpha_2''),$$

where the derivatives of f are evaluated at $(\alpha_1(t), \alpha_2(t))$, and α_1 etc. are evaluated at t. [6]

(iii) (a) Determine whether or not the curve $\alpha(t) = \mathbf{x}(t,t)$ is a geodesic of the surface parametrized by

$$\mathbf{x}(u,v) = (u, v, u^2 + v^2).$$ [4]

(b) Determine whether or not the curve $\alpha(t) = \mathbf{x}(t^2, t^2)$ is an asymptotic curve of the surface parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \tfrac{1}{2}(u^2 - v^2)\right).$$ [5]

You should be able to answer Questions 3–5 after you have completed your study of Part VI.

Question 3 (25 marks)

Let M be a surface parametrized by a single patch $\mathbf{x}$ for which $E = 1$ and $F = 0$.

(i) Define a tangent frame field on M in terms of the partial velocities. [4]

(ii) Find the dual 1-forms θ_1 and θ_2 for E_1 and E_2, expressing your answers in terms of du and dv. [4]

(iii) By using the first structural equations, show that

$$\omega_{12} = \frac{\partial}{\partial u}\left(\sqrt{G}\right) dv.$$ [6]

(iv) Use the second structural equation to show that

$$\frac{\partial^2}{\partial u^2}\left(\sqrt{G}\right) + K\sqrt{G} = 0.$$ [5]

(v) *Hence* find the Gaussian curvature function for the surface parametrized by

$$\mathbf{x}(u, v) = \left(\frac{u}{(1+v^2)^{1/2}}, v, \frac{uv}{(1+v^2)^{1/2}}\right).$$ [6]

Question 4 (20 marks)

A unit-speed curve α has Frenet apparatus T, N, B, $\kappa > 0$ and τ. The surface M is parametrized by the mapping $\mathbf{x}$ defined by

$$\mathbf{x}(u, v) = \alpha(u) + vT(u), \quad v > 0,$$

where u and v are further restricted sufficiently to ensure that $\mathbf{x}$ *is* the parametrization of a surface.

(i) Find expressions for the functions E, F and G. [5]

(ii) Express a unit normal vector field on M in terms of the Frenet apparatus of α. [4]

(iii) Find the Gaussian, mean and principal curvature functions for M. What can you deduce about the surface M? [4]

Let α be the unit-speed reparametrization of the curve defined by

$$\beta(t) = (\cosh t, \sinh t, t), \quad t \in \mathbf{R}.$$

(iv) Show that the corresponding surface cannot have planar points. [7]

Question 5 (10 marks)

The surface M is covered by a single orthogonal patch for which

$$E = \cosh^2 u, \quad G = 1, \quad F = 0.$$

By considering a suitable tangent frame field E_1, E_2, and the corresponding dual 1-forms θ_1, θ_2, show that M is flat. [10]

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M434 DIFFERENTIAL GEOMETRY 1993

study week	start date	course text	set book			ASSIGNMENT number	cut-off date
				There are no television, radio or audio programmes associated with this course.†			
0	Feb 6						
1	Feb 13	Part I, 1–3	I, 1–3				
2	Feb 20						
3	Feb 27	Part I, 4–5	I, 4, 5				
4	Mar 6						
5	Mar 13	Part I, 6–8	I, 6–8				
6	Mar 20						
7	Mar 27	Part II, 1–2	II, 1, 2				
8	Apr 3					TMA 01	Apr 6
	EASTER						
9	Apr 17	Part II, 3–5	II, 3–5				
10	Apr 24						
11	May 1	Part II, 6–8	II, 6–8				
12	May 8						
13	May 15	Part III, 1–3	III, 1–3				
14	May 22						
15	May 29	Part III, 4–5	III, 4, 5				
16	June 5					TMA 02	June 8

17	June 12	Part IV, 1–3	IV, 1–3				
18	June 19						
19	June 26	Part IV, 4–6	IV, 4–6				
20	July 3						
21	July 10	Part V, 1–2	V, 1, 2				
22	July 17						
23	July 24	Part V, 3	V, 3				
24	July 31					TMA 03	Aug 3
25	Aug 7	Part V, 4–5	V, 4, 5				
26	Aug 14						
27	Aug 21	Part V, 6 and Part VI, 1	V6, VI1				
28	Aug 28						
29	Sept 4	Part VI, 2–3	VI 2, 6				
30	Sept 11						
31	Sept 18	Part VI, 4–5					
32	Sept 25					TMA 04	Sept 28

Assessment strategy: Two essessment components: (i) TMAs 01–04 (50%); and (ii) the final examination (50%). Substitution as described in your Student Handbook will apply for up to one TMA.

Footnotes †There are, however, Faculty end generel University programmes which may be of interest to you.